Basis transformations in generation space and a criterion for the existence of standard forms for unitarily congruent matrices

1988 J. Phys. A: Math. Gen. 21 2825

(http://iopscience.iop.org/0305-4470/21/12/023)

J. Phys. A: Math. Gen. 21 (1988) 2825-2833. Printed in the UK

Basis transformations in generation space and a criterion for the existence of standard forms for unitarily congruent matrices[†]

Walter Grimus and Gerhard Ecker

Institut für Theoretische Physik, Universität Wien, Boltzmanngasse 5, A-1090 Wien, Austria

Received 12 January 1988, in final form 11 March 1988

Abstract. Basis transformations for fermion generations lead to a consideration of matrix transformations of the type U^TAU with unitary U. It is shown that A can be brought to a certain standard form iff A and A^T are simultaneously diagonalisable by a biunitary transformation. Application of this theorem allows for a standardisation of Yukawa couplings.

1. Motivation

The fermionic constituents of matter appear to be grouped in families or generations. In the framework of gauge theories, generations are equivalent representations of the gauge group under consideration. In general, a given generation consists of several inequivalent irreducible representations (irreps). For instance, in the standard model of electroweak interactions (Glashow 1961, Weinberg 1967, Salam 1968) a lepton family consists of a doublet and a singlet of SU(2), whereas a quark generation contains a doublet and two singlets, the singlets being distinguished by different U(1) quantum numbers. Although not really necessary for the following discussion, we restrict ourselves to the case of complete generations where each family has the same number of fermionic degrees of freedom.

The interactions of fermions with gauge fields are identical for each generation. In a gauge-invariant renormalisable Lagrangian quantum field theory the generation symmetry can only be broken by Yukawa couplings to scalar fields. Taking for convenience of notation all fermions as left-handed Weyl spinor fields, the Yukawa interaction has the general form

$$\mathscr{L}_Y = \psi^T_{ai} C^{-1} \Gamma^{ij}_{r,ab} \psi_{bj} \Phi^r + \text{HC} \qquad 1 \leqslant i, j \leqslant n_G. \tag{1.1}$$

The multiplet Φ contains all scalar fields in the theory, C is the Dirac charge conjugation matrix, n_G denotes the number of generations and a, b, r label the gauge group representations. Because of Fermi statistics the Yukawa couplings satisfy the symmetry relation

$$\Gamma^{ij}_{r,ab} = \Gamma^{ji}_{r,ba}. \tag{1.2}$$

[†] Supported in part by Jubiläumsfonds der Österreichischen Nationalbank, project no 2765.

A basis transformation in generation space consists of a set of unitary transformations acting on the generations, one for each irrep in a family. The gauge-invariant kinetic terms of the fermions are invariant under such a transformation, but the Yukawa Lagrangian (1.1) is modified. The Yukawa couplings connecting two irreps $\psi^{(1)}_{ai}$, $\psi^{(2)}_{bj}$ are transformed into

$$\hat{\Gamma}^{ij}_{r,ab} = U^{(1)}_{ki}\Gamma^{kl}_{r,ab}U^{(2)}_{lj} \tag{1.3}$$

where $U^{(1)}$, $U^{(2)}$ are the unitary basis transformations for $\psi^{(1)}$ and $\psi^{(2)}$, respectively. Specialising to the case $\psi^{(1)} \equiv \psi^{(2)}$, we find that (1.3) is of the form

$$\hat{A} = U^{\mathrm{T}}AU \tag{1.4}$$

with unitary U, i.e. the matrices A and $\hat{A}$ are unitarily congruent to each other[†].

The same unitary congruence arises in the discussion of generalised *CP* and *T* transformations (Ecker *et al* 1987) where A is now unitary. The example of time reversal suggests another application of (1.4): it governs, in fact, the transformation of a general antilinear mapping A under a change of basis implemented by U.

Unitary congruence occurs much less frequently both in physics and mathematics than unitary similarity

$$A' = U^{\dagger}AU. \tag{1.5}$$

The purpose of the present investigation is to establish an analogue to the well known theorem that A can be diagonalised by a unitary similarity transformation (1.5) iff A is normal. For the case of unitary congruence, only partial results seem to be known.

(i) If A is symmetric it can be brought to diagonal, positive semi-definite form (Schur 1945).

(ii) A unitary matrix A can be block-diagonalised by a congruence transformation (1.4) where each block is either a real orthogonal 2×2 matrix or the unit matrix of arbitrary dimension (Ecker *et al* 1987).

The generalisation of these results to be formulated as a theorem in § 2 will involve the notion of simultaneous diagonalisability of A and A^{T} through a biunitary transformation. In general, a set of square matrices $\{A_1, \ldots, A_N\}$ is said to be simultaneously diagonalisable by a biunitary transformation if there exist unitary matrices U, V such that $U^{\dagger}A_iV$ are diagonal for all $i = 1, \ldots, N$. At first sight, the requirement of simultaneous diagonalisability of A and A^{T} seems to be much more difficult to verify than the corresponding requirement of normality of A in the case of similarity. However, the task is greatly facilitated by the following theorem (Sartori 1979, Gatto *et al* 1980, Grimus and Ecker 1986): the set $\{A_1, \ldots, A_N\}$ is simultaneously diagonalisable by a biunitary transformation iff the sets $S_1 = \{A_i^{\dagger}A_j\}_{i,j=1,\ldots,N}$ and $S_2 = \{A_iA_j^{\dagger}\}_{i,j=1,\ldots,N}$ are Abelian. If at least one of the matrices A_i is non-singular it is actually sufficient to check if either S_1 or S_2 are Abelian (Grimus and Ecker 1986). In the present case of simultaneously diagonalisable A and A^{T} the sets S_1 and S_2 are complex conjugates of each other so it is again sufficient to investigate either one of them.

The previously known cases of symmetric or unitary A come as special cases under the general requirement of simultaneous bidiagonalisability of A and A^{T}. This is obvious for $A^{\mathrm{T}} = A$ because any matrix can be biunitarily diagonalised. It is also true for $A^{\dagger} = A^{-1}$ because any two unitary matrices are simultaneously bidiagonalisable as a straightforward application of Sartori's theorem shows. The basic assumption is also

---

[†] All matrices in this paper are square matrices over the field of complex numbers.

fulfilled for an antisymmetric matrix A, which will be relevant for the discussion of Yukawa couplings in § 3.

Independently of these special applications, we want to emphasise once again that the theorem of § 2 can be viewed as the analogue of the unitary diagonalisability of normal linear mappings in the case of antilinear mappings.

2. Unitary congruence and standard forms

Theorem. Let A be a complex $n \times n$ matrix. Then A and A^{T} are simultaneously diagonalisable by a biunitary transformation iff there exists a unitary $n \times n$ matrix U such that

$$U^{\mathrm{T}} A U = \text{block-diag}(B_1, \dots, B_k, C) \tag{2.1}$$

with 2×2 matrices $B_i (i = 1, \dots, k)$ of the form

$$B_i = \begin{pmatrix} 0 & a_i \\ e^{i\varphi_i} b_i & 0 \end{pmatrix} \tag{2.2}$$

$$a_i > 0 \qquad b_i \geq 0 \qquad 0 \leq \varphi_i \leq \pi \qquad a_i \neq b_i \text{ for } \varphi_i = 0$$

and with a positive semi-definite diagonal $l \times l$ matrix $C = \operatorname{diag}(c_1, \dots, c_l)$.

Before proving the theorem we want to make a few remarks.

(i) The real numbers a_i, b_i, φ_i and c_j can be obtained from the eigenvalues

$$a_i^2, b_i^2 \quad (i = 1, \dots, k) \qquad c_j^2 \quad (j = 1, \dots, l)$$

of $AA^\dagger$ and from the eigenvalues

$$a_i b_i e^{\pm i\varphi_i} \quad (i = 1, \dots, k) \qquad c_j^2 \quad (j = 1, \dots, l)$$

of AA^*. Note that in view of Sartori's theorem the simultaneous bidiagonalisability of A and A^{T} implies that AA^* is normal and therefore diagonalisable by a unitary similarity transformation.

(ii) For $a_i = b_i$ an equivalent standard form for B_i is given by

$$\tilde{U}_i^{\mathrm{T}} B_i \tilde{U}_i = a_i \begin{pmatrix} \cos \frac{1}{2}\varphi_i & \sin \frac{1}{2}\varphi_i \\ -\sin \frac{1}{2}\varphi_i & \cos \frac{1}{2}\varphi_i \end{pmatrix} \tag{2.3}$$

with

$$\tilde{U}_i = \frac{e^{-i(\pi + \varphi_i)/4}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}. \tag{2.4}$$

In particular, this case is realised for unitary A. From (2.3) it is also clear that for $a_i = b_i$ and $\varphi_i = 0$ the matrix B_i can be diagonalised and put into C. This explains the exclusion in (2.2).

Proof. Multiplying (2.1) and the transposed equation by a block-diagonal unitary matrix either from the left or from the right, with k blocks of two-dimensional permutation matrices and an l-dimensional unit matrix, immediately shows that A and A^{T} are simultaneously diagonalisable by a biunitary transformation.

Proving the opposite implication is more involved. By assumption, there exist unitary matrices V, W such that

$$W^\dagger A V = D \tag{2.5a}$$

$$W^\dagger A^{\mathrm{T}} V = D' \tag{2.5b}$$

with diagonal D, D'. Without loss of generality, D can be taken real and positive semi-definite. Defining a unitary matrix

$$S = V^{\mathrm{T}} W \tag{2.6}$$

one easily derives the relations

$$D' = SDS^* \tag{2.7a}$$

$$DD'^* = S^\dagger DD'S. \tag{2.7b}$$

Consequently, for every eigenvalue λ of DD' also λ^* is an eigenvalue. The same is, of course, valid for $AA^* = WDD'^*W^\dagger$.

Assuming p different complex eigenvalues λ_α $(\alpha = 1, \ldots, p)$ of AA^* with $0 < \arg\lambda_\alpha =: \psi_\alpha < \pi$ and q real eigenvalues $\lambda_\alpha (\alpha = p+1, \ldots, p+q = n-p)$ we can write

$$DD'^* = \operatorname{diag}(\lambda_1 \mathbf{1}_{m_1}, \lambda_1^* \mathbf{1}_{m_1}, \ldots, \lambda_p \mathbf{1}_{m_p}, \lambda_p^* \mathbf{1}_{m_p}, \lambda_{p+1} \mathbf{1}_{m_{p+1}}, \ldots, \lambda_{p+q} \mathbf{1}_{m_{p+q}}) \tag{2.8}$$

with m_α the multiplicity of λ_α and $\mathbf{1}_{m_\alpha}$ the m_α-dimensional unit matrix. From (2.7b) and (2.8) it follows that S has the form

$$S = \begin{pmatrix} 0 & S_1 & & & & & & \\ S_1' & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & S_p & & & \\ & & & S_p' & 0 & & & \\ & & & & & S_{p+1} & & \\ & & & & & & \ddots & \\ & & & & & & & S_{p+q} \end{pmatrix} \tag{2.9}$$

with all S_α and S_α' being $m_\alpha \times m_\alpha$ unitary matrices.

Expressing the λ_α by D and S and removing V from (2.5a) we get

$$DD'^* = DS^*DS \tag{2.10}$$

and

$$W^\dagger A W^* = DS^*. \tag{2.11}$$

These two equations will now be discussed separately for each λ_α in view of (2.8) and (2.9).

Let us first consider a complex λ_α. In this case (2.10) is

$$\begin{pmatrix} \lambda_\alpha \mathbf{1}_{m_\alpha} & 0 \\ 0 & \lambda_\alpha^* \mathbf{1}_{m_\alpha} \end{pmatrix} = D_\alpha \begin{pmatrix} 0 & S_\alpha^* \\ S_\alpha'^* & 0 \end{pmatrix} D_\alpha \begin{pmatrix} 0 & S_\alpha \\ S_\alpha' & 0 \end{pmatrix} \tag{2.12}$$

where D_α is the corresponding part of D. With the notation

$$D_\alpha = \sqrt{|\lambda_\alpha|} \begin{pmatrix} E_\alpha & 0 \\ 0 & F_\alpha \end{pmatrix} \tag{2.13}$$

with diagonal and positive E_α, F_α, (2.12) can be written in the form†

$$S'_\alpha = \mathrm{e}^{\mathrm{i}\psi_\alpha} F_\alpha S_\alpha^{\mathrm{T}} E_\alpha \tag{2.14}$$

$$S_\alpha = \mathrm{e}^{-\mathrm{i}\psi_\alpha} E_\alpha S_\alpha'^{\mathrm{T}} F_\alpha . \tag{2.15}$$

Eliminating S'_α leads to the relation

$$S_\alpha = E_\alpha^2 S_\alpha F_\alpha^2 \tag{2.16}$$

so that E_α and F_α^{-1} have the same spectrum. By a permutation of basis vectors in the α sector we can always achieve

$$F_\alpha = E_\alpha^{-1} . \tag{2.17}$$

Moreover, (2.16) implies

$$E_{\alpha,ii} F_{\alpha,jj} = 1 \tag{2.18}$$

for any pair of indices of i, j with $S_{\alpha,ij} \neq 0$. Equation (2.18) together with either (2.14) or (2.15) then give rise to the matrix equation

$$S'_\alpha = \mathrm{e}^{\mathrm{i}\psi_\alpha} S_\alpha^{\mathrm{T}} . \tag{2.19}$$

Therefore, we have

$$(DS^*)_\alpha := D_\alpha \begin{pmatrix} 0 & S_\alpha^* \\ S_\alpha'^* & 0 \end{pmatrix} = \sqrt{|\lambda_\alpha|} \begin{pmatrix} E_\alpha & 0 \\ 0 & E_\alpha^{-1} \end{pmatrix} \begin{pmatrix} 0 & S_\alpha^* \\ \mathrm{e}^{-\mathrm{i}\psi_\alpha} S_\alpha^\dagger & 0 \end{pmatrix} . \tag{2.20}$$

We can now perform a transformation of the required type with

$$U_\alpha = \mathrm{e}^{\mathrm{i}\psi_\alpha/2} \begin{pmatrix} 0 & \mathbf{1}_{m_\alpha} \\ S_\alpha^T & 0 \end{pmatrix} \tag{2.21}$$

to obtain finally

$$U_\alpha^{\mathrm{T}} (DS^*)_\alpha U_\alpha = \sqrt{|\lambda_\alpha|} \begin{pmatrix} 0 & E_\alpha^{-1} \\ \mathrm{e}^{\mathrm{i}\psi_\alpha} E_\alpha & 0 \end{pmatrix} . \tag{2.22}$$

Keeping in mind equation (2.11), the last equation is already of the desired form (2.1) and (2.2) up to a trivial rearrangement of the basis. The a_i are to be identified with $\sqrt{|\lambda_\alpha|} E_\alpha^{-1}$ and the complex numbers $\mathrm{e}^{\mathrm{i}\varphi_i} b_i$ with $\mathrm{e}^{\mathrm{i}\psi_\alpha} \sqrt{|\lambda_\alpha|} E_\alpha$ completing the proof for complex eigenvalues of AA^*.

For a negative eigenvalue λ_α (2.10) yields

$$S_\alpha^{\mathrm{T}} = -D_\alpha S_\alpha D_\alpha / |\lambda_\alpha| . \tag{2.23}$$

With similar arguments as in the case of complex λ_α, (2.23) is shown to imply

$$S_\alpha^{\mathrm{T}} = -S_\alpha \tag{2.24}$$

and

$$S_\alpha^\dagger \frac{D_\alpha}{\sqrt{|\lambda_\alpha|}} S_\alpha = \left(\frac{D_\alpha}{\sqrt{|\lambda_\alpha|}} \right)^{-1} . \tag{2.25}$$

---

† Actually, (2.14) and (2.15) can be shown to be equivalent.

Since S_α is non-singular the multiplicity of a negative eigenvalue of AA^* must be even. Furthermore, D_α can be written as

$$D_\alpha = \sqrt{|\lambda_\alpha|}\ \mathrm{diag}(\mathbf{1}_{\mu_{0\alpha}}, d_{1\alpha}\mathbf{1}_{\mu_{1\alpha}}, d_{1\alpha}^{-1}\mathbf{1}_{\mu_{1\alpha}}, \ldots) \qquad \mu_{0\alpha}\ \text{even} \qquad d_{i\alpha} > 1 \tag{2.26}$$

and S_α must have the form

$$S_\alpha = \begin{pmatrix} T_{0\alpha} & & & \\ & 0 & T_{1\alpha} & \\ & -T_{1\alpha}^{\mathrm{T}} & 0 & \\ & & & \ddots \end{pmatrix} \qquad T_{0\alpha}^{\mathrm{T}} = -T_{0\alpha}. \tag{2.27}$$

Applying lemma 1 of the appendix to the matrix $T_{0\alpha}$ we can find a unitary matrix t_α such that

$$t_\alpha^{\mathrm{T}} T_{0\alpha} t_\alpha = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}. \tag{2.28}$$

Thus, we finally arrive at

$$U_\alpha^{\mathrm{T}}(DS^*)_\alpha U_\alpha = \sqrt{|\lambda_\alpha|} \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & d_{1\alpha}\mathbf{1}_{\mu_{1\alpha}} & \\ & & & -d_{1\alpha}^{-1}\mathbf{1}_{\mu_{1\alpha}} & 0 & \\ & & & & & \ddots \end{pmatrix} \tag{2.29}$$

with

$$U_\alpha = \text{block-diag}(t_\alpha^*, 1_{\mu_{1\alpha}}, T_{1\alpha}^{\mathrm{T}}, \ldots). \tag{2.30}$$

Rearranging again some basis vectors, (2.29) can be brought into the form maintained in the theorem.

In the case of $\lambda_\alpha = 0$ equation (2.10) reads

$$D_\alpha S_\alpha D_\alpha = 0 \tag{2.31}$$

implying $\det D_\alpha = 0$. Writing

$$D_\alpha = \begin{pmatrix} \tilde{D}_\alpha & 0 \\ 0 & 0 \end{pmatrix} \tag{2.32}$$

with a positive r-dimensional diagonal matrix $\tilde{D}_\alpha$, one finds that S_α must have the form

$$S_\alpha = \begin{pmatrix} 0 & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \tag{2.33}$$

with an $r \times r$ zero matrix and $r \times s$, $s \times r$, $s \times s$ matrices S_{12}, S_{21}, S_{22}, respectively $(r + s = m_\alpha)$. Because of the unitarity of S_α the rows of S_{12} form an orthonormal set of r s-dimensional vectors so that necessarily $r \leqslant s$. Denoting the rows of S_{12} as

$v_1^{\mathrm{T}}, \ldots, v_r^{\mathrm{T}}$ with column vectors v_β ($\beta = 1, \ldots, r$) we can find vectors $v_{r+1}, \ldots, v_s$ to make $\{v_1, \ldots, v_s\}$ an orthonormal basis of $\mathbf{C}^s$. Then the matrix

$$T = \begin{pmatrix} 1_r & 0 \\ 0 & (v_1, \ldots, v_s) \end{pmatrix} \tag{2.34}$$

is unitary and

$$T^{\mathrm{T}}(DS^*)_\alpha T = \begin{pmatrix} 0 & (\tilde{D}_\alpha, 0) \\ 0 & 0 \end{pmatrix} \tag{2.35}$$

with $(\tilde{D}_\alpha, 0)$ forming an $r \times s$ matrix. In this case, there are r matrices B_i in (2.1) and (2.2) with $b_i = 0$ and a zero in the diagonal matrix C with multiplicity $s - r$.

The remaining case $\lambda_\alpha > 0$ can be discussed in close analogy to $\lambda_\alpha < 0$. The corresponding matrix $T_{0\alpha}$ is now symmetric. Lemma 2 of the appendix furnishes a unitary symmetric matrix t_α with $t_\alpha T_{0\alpha} t_\alpha = 1_{\mu_{0\alpha}}$. Therefore, the $\mu_{0\alpha}$ elements $\sqrt{\lambda_\alpha}$ of D_α contribute to the diagonal matrix C whereas the rest yields matrices B_i with $\varphi_i = 0$. This concludes the proof of the theorem.

3. Yukawa couplings

Following the original motivation in § 1, we shall now apply the theorem of the previous section to the Yukawa couplings of a given scalar irrep to a corresponding irreducible fermionic bilinear. More precisely, we consider a certain irreducible part of the total Yukawa Lagrangian of the form

$$\mathscr{L}_{\mathrm{Y}}^{\mathrm{irr}} = \psi_{ai}^{(1)\mathrm{T}} \Gamma_{r,ab}^{ij} \psi_{bj}^{(2)} \Phi^r + \mathrm{HC} \tag{3.1}$$

where $\psi_i^{(1)}$, $\psi_j^{(2)}$ and Φ are all irreps, the fermionic irreps being identical for all generation indices i and j, respectively. Moreover, $\psi_i^{(1)}$ and $\psi_j^{(2)}$ are assumed to give rise to the same irreducible fermionic bilinear for all i, j. In this case

$$\Gamma_{r,ab}^{ij} = c_{r,ab} \gamma_{ij} \tag{3.2}$$

where $c_{r,ab}$ are the Clebsch–Gordan coefficients for projecting† out of $\psi^{(1)} \times \psi^{(2)}$ the irrep complex conjugate to Φ. In many cases, the relation (3.2) is automatically satisfied. Only when Φ^* appears more than once in the Kronecker product $\psi^{(1)} \times \psi^{(2)}$ the additional assumption is needed.

We can now formulate the following proposition for canonical Yukawa couplings.

Proposition. For an irreducible Yukawa Lagrangian (3.1) one can always choose a basis in generation space such that the Yukawa coupling matrix γ defined in (3.2) assumes a certain real standard form. Three cases must be distinguished.

(i) For $\psi^{(1)} \neq \psi^{(2)}$, γ can be made diagonal and positive semi-definite.

(ii) If $\psi^{(1)} = \psi^{(2)}$ and if Φ^* is in the symmetric Kronecker product $(\psi \times \psi)_{\mathrm{S}}$, γ can again be made diagonal and positive semi-definite.

---

† We use the same letters for fields and irreps.

(iii) With $\psi^{(1)}=\psi^{(2)}$ and Φ^* contained in the antisymmetric Kronecker product $(\psi\times\psi)_{\mathrm{A}}$, γ can be transformed to block-diagonal form

$$\gamma=\text{block-diag}(B_1,\ldots,B_k,0) \tag{3.3}$$

with

$$B_i=\begin{pmatrix}0 & b_i\\ -b_i & 0\end{pmatrix} \qquad b_i\in\boldsymbol{R}.$$

Case (i) is due to the fact that any complex matrix can be brought to diagonal, positive semi-definite form by a biunitary transformation. Parts (ii) and (iii) are immediate consequences of the theorem of § 2, recalling (1.2) and the symmetry or antisymmetry in a, b, respectively of the Clebsch-Gordan coefficients $c_{r,ab}$.

The above proposition implies in particular that all phases in the Yukawa couplings γ_{ij} can be rotated away for an irreducible Yukawa Lagrangian (3.1). Thus, the *CP* properties of $\mathscr{L}_{\mathrm{Y}}^{\mathrm{irr}}$ are determined by the Clebsch-Gordan coefficients $c_{r,ab}$ only. Of course, the complete Yukawa Lagrangian will not be irreducible in general.

Canonical Yukawa couplings may be of advantage to determine the number of relevant parameters of the theory. The freedom of performing basis transformations in generation space prior to spontaneous symmetry breaking introduces redundant parameters without physical significance. As an illustrative example, we consider an $\mathrm{SU}(2)_L\times\mathrm{U}(1)$ model for neutrino masses (Zee 1980) where the lepton doublets interact with a singlet scalar field. Coupling two doublets to a singlet requires antisymmetric Clebsch-Gordan coefficients and thus antisymmetric Yukawa couplings γ_{ij}. Referring to (3.3), we observe that of the original $n_{\mathrm{G}}(n_{\mathrm{G}}-1)/2$ complex Yukawa couplings only $[n_{\mathrm{G}}/2]$ real parameters remain in a canonical basis where $[n_{\mathrm{G}}/2]$ stands for the largest integer not exceeding $n_{\mathrm{G}}/2$. For instance, for $n_{\mathrm{G}}=3$ the three complex couplings are reduced to a single real parameter. More generally in the case of antisymmetric Yukawa couplings, one can always find a basis for n_{G} odd where at least one generation decouples. Moreover, the canonical form of Yukawa couplings makes it rather easy to determine all remaining basis transformations leaving the canonical form unchanged. Taking again the Zee model for $n_{\mathrm{G}}=3$ as an example, the Yukawa coupling matrix

$$\gamma=\begin{pmatrix}0 & a & 0\\ -a & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{3.4}$$

is unchanged iff the basis transformation is of the form

$$U=\begin{pmatrix}\mathrm{e}^{\mathrm{i}\alpha}\cos\vartheta & \mathrm{e}^{\mathrm{i}\beta}\sin\vartheta & 0\\ -\mathrm{e}^{-\mathrm{i}\beta}\sin\vartheta & \mathrm{e}^{-\mathrm{i}\alpha}\cos\vartheta & 0\\ 0 & 0 & \mathrm{e}^{\mathrm{i}\gamma}\end{pmatrix}. \tag{3.5}$$

In other words, the two generations coupling to the singlet scalar field can still be subjected to an almost arbitrary unitary transformation. This rotation can be used to reduce the number of relevant parameters in the Yukawa couplings of other scalar fields.

Acknowledgment

We thank H Neufeld for helpful discussions.

Appendix

Lemma 1. Let S be a unitary antisymmetric $n\times n$ matrix. Then there exists a unitary matrix T such that

$$T^{\mathrm{T}}ST=\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}. \tag{A1}$$

Proof. S can be written as $S=S_1+\mathrm{i}S_2$ with real and antisymmetric S_i. From unitarity we have

$$\mathbf{1}=S^{\dagger}S=-S_1^2-S_2^2-i[S_1,S_2]. \tag{A2}$$

Consequently $[S_1,S_2]=0$ and there is an orthogonal matrix O such that

$$O^{\mathrm{T}}S_iO=\begin{pmatrix} 0 & a_1^{(i)} & & & \\ -a_1^{(i)} & 0 & & & \\ & & \ddots & & \\ & & & 0 & a_{n/2}^{(i)} \\ & & & -a_{n/2}^{(i)} & 0 \end{pmatrix} \tag{A3}$$

for both $i=1$, 2 (n must be even!). Since S is unitary we obtain

$$|a_\alpha^{(1)}+\mathrm{i}a_\alpha^{(2)}|=1 \qquad (\alpha=1,\ldots,n/2). \tag{A4}$$

Therefore, there exists a diagonal phase matrix P such that $T=OP$.

Lemma 2. Let S be a unitary symmetric matrix. Then there exists a unitary symmetric matrix T with $S=T^2$.

Proof. We can write $S=S_1+\mathrm{i}S_2$ with S_i real and symmetric. As before, it follows from unitarity that S_1 and S_2 commute and can therefore be diagonalised simultaneously by an orthogonal matrix O. Then $P:=O^{\mathrm{T}}SO$ is a diagonal phase matrix and T is obtained as $T=O\sqrt{P}O^{\mathrm{T}}$.

References

Ecker G, Grimus W and Neufeld H 1987 *J. Phys. A: Math. Gen.* **20** L807
Gatto R, Morchio G, Sartori G and Strocchi F 1980 *Nucl. Phys.* B **163** 221
Glashow S L 1961 *Nucl. Phys.* **22** 579
Grimus W and Ecker G 1986 *J. Phys. A: Math. Gen.* **19** 3917
Salam A 1968 *Proc. 8th Nobel Symp., Aspenäsgården, 1968* ed N Svartholm (Stockholm: Almqvist and Wiksell)
Sartori G 1979 *Phys. Lett.* **82B** 255
Schur I 1945 *Am. J. Math.* **67** 472
Weinberg S 1967 *Phys. Rev. Lett.* **19** 1264
Zee A 1980 *Phys. Lett.* **93B** 389